

Prepotential approach to exact and quasi-exact solvabilities

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Exact and quasi-exact solvabilities of the one-dimensional Schrödinger equation are discussed from a unified viewpoint based on the prepotential together with Bethe ansatz equations. This is a constructive approach which gives the potential as well as the eigenfunctions and eigenvalues simultaneously. The novel feature of the present work is the realization that both exact and quasi-exact solvabilities can be solely classified by two integers, the degrees of two polynomials which determine the change of variable and the zero-th order prepotential. Most of the well-known exactly and quasi-exactly solvable models, and many new quasi-exactly solvable ones, can be generated by appropriately choosing the two polynomials. This approach can be easily extended to the constructions of exactly and quasi-exactly solvable Dirac, Pauli, and Fokker-Planck equations.

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I. INTRODUCTION

Two decades ago, a new class of potentials which are intermediate to exactly solvable (ES) potentials and non solvable ones have been found for the Schrödinger equation. These are called quasi-exactly solvable (QES) models for which it is possible to determine algebraically a part of the spectrum but not the whole spectrum [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The discovery of this class of spectral problems has greatly enlarged the number of physical systems which we can study analytically. In the last few years, QES theory has also been extended to the Pauli [14] and Dirac equations [15, 16, 17, 18, 19, 20, 21, 22, 23]. More recently, we have considered QES quasinormal modes, which are damping modes with complex eigen-energies [24]. Such modes are of interest in black hole physics.

Usually a QES problem admits a certain underlying Lie algebraic symmetry which is responsible for the quasi-exact solutions. Such underlying symmetry is most easily studied in the Lie-algebraic approach [3, 6, 7, 8, 9, 10, 11, 12, 13]. However, solutions of QES states are more directly found in the analytic approach based on the Bethe ansatz equations [4, 5, 25]. In this analytic approach the form of the wave functions containing some parameters are assumed from the very beginning, and these parameters are fitted to make the ansatz compatible with the potential under consideration. Further developments in QES theory include classification of one-dimensional QES operators possessing finite-dimensional invariant subspace with a basis of monomials [26], and formulation extending to nonlinear operators [27].

A different direction in the development of QES theory is the prepotential approach [28]. Here the emphasis was shifted from the potential to the so-called prepotential (or superpotential), a concept which plays a fundamental role in supersymmetric quantum mechanics [29, 30] (we mention here that the prepotential W considered here is the integral of the superpotential W in [14, 21, 22, 29, 30]). Prepotential has been extensively employed to study classical and quantum integrability in Calogero-Moser systems [31, 32, 33, 34, 35, 36]. The simplest prepotentials are those which give rise to QES potentials admitting just the ground states. Physically these are factors which take care of the asymptotic behaviors of the systems. They are called the gauge factors in [2, 3] (which we shall call the zero-th order prepotential). Classification of all possible gauge factors for $sl(2)$ -based QES systems is presented in [7]. Unlike previous works, however, in the prepotential approach of [28] the prepotential assumes a more fundamental role. From a different consideration, it was found that the Schrödinger QES theory was most easily extended to systems with multi-component wave functions by recasting the Lie-algebraic theory in terms of the prepotentials [14, 21, 22]. More recently, QES theory was extended to the Fokker-Planck equations also via the prepotential approach [37].

It thus seems that the prepotential approach to QES theory deserves a more in-depth study than it has received so far. The merit of this approach is that the form of the potential of the system concerned needs not be assumed from the beginning. All information about the system is contained in the prepotential and the solutions, or roots, of the Bethe ansatz equations. The prepotential and the roots determine the potential as well as the eigenfunctions and eigenvalues simultaneously. Also, in this approach exact and quasi-exact solvabilities can be treated on the same footing. Furthermore, as mentioned in the last paragraph, such approach facilitates extension of the QES theory from the Schrödinger equation to equations for multi-component wave functions.

The ideas of the prepotential approach to exact and quasi-exact solvabilities have been presented in [14, 21, 22, 28], and summarized in [37]. The emphasis of these works was placed on the feasibility and elegance of the prepotential approach. However, the forms of the prepotential and the required change of coordinates were either directly adapted from the known ES and QES models, or given as known for the new QES systems. Now we would like to give a first attempt to address the questions as to how the choice of coordinate transformation and the prepotential

are determined, at least for certain classes of coordinates and prepotentials.

The main result of this work is the realization that exact and quasi-exact solvabilities can be solely classified by two integers, the degrees of the two polynomials which determine the change of variable and the zero-th order prepotential. Such classification scheme has not been explored before. There are upper limits for these two integers beyond which no ES systems are possible. By selecting appropriate values of these two degrees, most of the well-known ES and QES models, and many new QES ones, are easily generated. This approach does not rely on the knowledge of possible existence of any underlying symmetry in the system concerned. It treats both exact and quasi-exact solvabilities on the same footing.

This paper is organized as follows. In Sect. II we present the idea of the prepotential approach to the exact/quasi-exact solvability of the Schrödinger equation. The general conditions which the required change of variable and the choice of the prepotential are discussed. Two common types of transformations of coordinates leading to exact/quasi-exact solvable systems are discussed and examples presented in Sect. III and IV. In Sect. V and VI the prepotential is generalized to systems defined on the half-line and on finite interval, respectively. Sect. VII concludes the paper.

II. PREPOTENTIAL APPROACH

Suppose $\phi_0(x)$ ($-\infty < x < \infty$) is the ground state, with zero energy, of a Hamiltonian H_0 : $H_0\phi_0 = 0$. By the well known oscillation theorem ϕ_0 is nodeless, and thus can be written as $\phi_0 \equiv e^{-W_0(x)}$, where $W_0(x)$ is a regular function of x (this will be assumed in the rest of this paper). For the square-integrable ϕ_0 , this is the simplest example of quasi-exact solvability. This implies that the potential V_0 is completely determined by W_0 : $V_0 = W_0'^2 - W_0''$, and consequently, the Hamiltonian is factorizable (we adopt the unit system in which \hbar and the mass m of the particle are such that $\hbar = 2m = 1$):

$$H_0 = \left(-\frac{d}{dx} + W_0'\right) \left(\frac{d}{dx} + W_0'\right). \quad (1)$$

This fact can be considered as the very base of the factorization method [38, 39] and of supersymmetric quantum mechanics [29, 30]. We shall call $W_0(x)$ the zero-th order prepotential.

Consider now a wave function ϕ_N ($N \geq 0$) which is related to ϕ_0 of H_0 by $\phi_N = \phi_0 \tilde{\phi}_N$, where

$$\tilde{\phi}_N = (z - z_1)(z - z_2) \cdots (z - z_N), \quad \tilde{\phi}_0 \equiv 1. \quad (2)$$

Here $z = z(x)$ is some function of x . In taking the form of $\tilde{\phi}_N$ in Eq. (2) we have assumed that the only singularities of the system are $z = \pm\infty$. Other situations will be addressed later. The function $\tilde{\phi}_N$ is a polynomial in an $(N+1)$ -dimensional Hilbert space with the basis $\langle 1, z, z^2, \dots, z^N \rangle$. One can rewrite ϕ_N as

$$\phi_N = \exp(-W_N(x, \{z_k\})), \quad (3)$$

with the N -th order prepotential W_N being defined by

$$W_N(x, \{z_k\}) = W_0(x) - \sum_{k=1}^N \ln |z(x) - z_k|. \quad (4)$$

Operating on ϕ_N by the operator $-d^2/dx^2$ results in a Schrödinger equation $H_N\phi_N = 0$, where

$$H_N = -\frac{d^2}{dx^2} + V_N, \quad (5)$$

$$V_N \equiv W_N'^2 - W_N''. \quad (6)$$

From Eq. (4), V_N has the form $V_N = V_0 + \Delta V_N$, where $V_0 = W_0'^2 - W_0''$, and

$$\Delta V_N \equiv -2 \left(W_0' z' - \frac{z''}{2} \right) \sum_{k=1}^N \frac{1}{z - z_k} + \sum_{\substack{k,l \\ k \neq l}}^N \frac{z'^2}{(z - z_k)(z - z_l)}. \quad (7)$$

Here the prime denotes derivative w.r.t. the variable x .

ΔV_N is generally a meromorphic function of z with at most simple poles. Let us demand that the residues of the simple poles, z_k , $k = 1, \dots, N$ should all vanish. This will result in a set of algebraic equations which the parameters

$\{z_k\}$ must satisfy. These equations are called the Bethe ansatz equations for $\{z_k\}$. With $\{z_k\}$ satisfying the Bethe ansatz equations, ΔV_N will have no simple poles at $\{z_k\}$ but it still generally depends on $\{z_k\}$. Thus the form of V_N is determined by the choice of z'^2 and $W_0(x)$.

In what follows we would like to demonstrate that the choice of z'^2 and $W_0'z' = P_m(z)$, $z'^2 = Q_n(z)$ and $z'' = R_l(z)$ are polynomials in z of degree m , n and l , respectively. Of course, $Q_n(z)$ and $R_l(z)$ are not independent. In fact, from $2z'' = dz'^2/dz$ we have $l = n - 1$ and

$$z'' = R_{n-1}(z) = \frac{1}{2} \frac{dQ_n(z)}{dz}. \quad (8)$$

Consequently, the variables x and z are related by (we assume $z(x)$ is invertible for practical purposes)

$$x(z) = \pm \int^z \frac{dz}{\sqrt{Q_n(z)}}, \quad (9)$$

and the prepotential $W_0(x)$ is determined as

$$W_0(x) = \int^x dx \left(\frac{P_m(z)}{\sqrt{Q_n(z)}} \right)_{z=z(x)}. \quad (10)$$

Eqs. (9) and (10) define the transformation $z(x)$ and the corresponding prepotential $W_0(x)$. Thus, $P_m(z)$ and $Q_n(z)$ determine the quantum system. Of course, the choice of P_m and Q_n must be such that W_0 derived from (10) must ensure normalizability of $\phi_0 = \exp(-W_0)$.

Now depending on the degrees of the polynomials P_m and Q_n , we have the following situations:

- (i) if $\max\{m, n-1\} \leq 1$, then in $V_N(x)$ the parameter N and the roots z_k 's will only appear as an additive constant and not in any term involving powers of z . Such system is then exactly solvable;
- (ii) if $\max\{m, n-1\} = 2$, then N will appear in the first power term in z , but z_k 's only in an additive term. This system then belongs to the so-called type 1 QES system defined in [3], i.e., for each $N \geq 0$, V_N admits $N+1$ solvable states with the eigenvalues being given by the $N+1$ sets of roots z_k 's. This is the main type of QES systems considered in the literature;
- (iii) if $\min\{m, n-1\} \geq 3$, then not only N but also z_k 's will appear in terms involving powers of z . This means that for each $N \geq 0$, there are $N+1$ different potentials V_N , differing in several parameters in terms involving powers of z , have the same eigenvalue (when the additive constant, or the zero point, is appropriately adjusted). When z_k 's appear only in the first power term in z , such systems are called type 2 QES systems in [3]. We see that QES models of higher types are possible.

We will illustrate these general situations with specific examples in the following sections. For definiteness, in this paper we will only consider cases with $z'^2 = Q_2(z) \equiv q_2 z^2 + q_1 z + q_0$ (i.e., $z'' = q_2 z + q_1/2$), where q_2 , q_1 and q_0 are real constants (by taking $n = 2$ here we mean to include Q_1 and Q_0 as special cases if some of the coefficients vanish). This choice of z'^2 covers most of the known ES shape-invariant potentials in [29, 30] and the $sl(2)$ -based QES systems in [3], and a new one discussed in [28, 37]. Such coordinates are called ‘‘sinusoidal coordinates’’, which include quadratic polynomials, trigonometric, hyperbolic, and exponential types. The connection of sinusoidal coordinates with ES theory has been extensively discussed in [40].

With this choice of z'^2 , we have

$$V_N = W_0'^2 - W_0'' + q_2 N^2 - 2 \sum_{k=1}^N \frac{1}{z - z_k} \left\{ P_m(z) - \frac{q_2}{2} z_k - \frac{q_1}{4} - \sum_{l \neq k} \frac{Q_2(z_k)}{z_k - z_l} \right\}. \quad (11)$$

In deriving Eq. (11) use has been made of the following identities:

$$\sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{1}{(z - z_k)(z - z_l)} = 2 \sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{1}{z - z_k} \left(\frac{1}{z_k - z_l} \right), \quad (12)$$

$$\sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{z}{(z - z_k)(z - z_l)} = 2 \sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{1}{z - z_k} \left(\frac{z_k}{z_k - z_l} \right), \quad (13)$$

$$\sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{z^2}{(z-z_k)(z-z_l)} = 2 \sum_{\substack{k,l=1 \\ k \neq l}}^N \frac{1}{z-z_k} \left(\frac{z_k^2}{z_k-z_l} \right) + N(N-1). \quad (14)$$

Demanding the residues at z_k 's vanish gives the set of Bethe ansatz equations

$$P_m(z_k) - \frac{q_2}{2} z_k - \frac{q_1}{4} - \sum_{l \neq k} \frac{Q_2(z_k)}{z_k - z_l} = 0, \quad k = 1, 2, \dots, N. \quad (15)$$

Putting back the set of roots z_k into Eq. (11), we obtain a potential $V_N(x)$ without simple poles. The degree of $P_m(z)$ determines the nature of the solvability of the system, namely, for $m = 1, 2, 3, \dots$, the system is, respectively, ES, type 1 QES, type 2 QES, and higher types QES, as discussed generally in the last section.

To proceed further, we must specify $P_m(z)$. We shall discuss cases where z'^2 is linear and quadratic in z separately.

III. EXAMPLES WITH $z'^2(x) = Q_1(z)$

Let us first consider the case where $z'^2(x) = Q_1(z)$, i.e., $q_2 = 0$. For definiteness we take $Q_1(z) = 4Az + q_0$ ($q_1 = 4A$), where A, q_0 are real constants. This implies $z''(x) = R_0(z) = 2A$. Hence, the general solution of $z(x)$ is a quadratic form of x :

$$z(x) = Ax^2 + Bx + C, \quad A, B, C: \text{real constants}. \quad (16)$$

It is easily checked that q_0 is related to A, B and C through $z'^2 = 4Az + B^2 - 4AC$.

We now illustrate how some ES and QES models can be constructed in the prepotential approach by taking different values of m .

A. Exactly solvable cases: $m = 1$

Suppose $m = 1$ so that $P_1(z) = A_1z + A_0$ (A_1, A_0 real). By writing $P_1 = A_1(z - z_k) + A_1z_k + A_0$, one obtains from Eq. (11) that

$$\begin{aligned} \Delta V_N &= -2A_1 \sum_{k=1}^N 1 - 2 \sum_{k=1}^N \frac{1}{z - z_k} \left\{ P_1(z_k) - A - \sum_{l \neq k} \frac{Q_1(z_k)}{z_k - z_l} \right\} \\ &= -2A_1 N. \end{aligned} \quad (17)$$

The last term in braces in Eq. (17) vanishes when z_k 's satisfy the Bethe ansatz equations (15). Now N only appears as a parameter in an additive term, and not in terms involving powers of z in ΔV_N . The roots z_k 's do not appear at all. The additive term can be treated as the eigenvalue. The Schrödinger equation reads

$$\left(-\frac{d^2}{dx^2} + W_0'^2 - W_0'' \right) e^{-W_N} = 2A_1 N e^{-W_N}. \quad (18)$$

We see that the potential $W_0'^2 - W_0''$ is ES: by varying N , one obtains all the eigenvalues $2A_1 N$ and the eigenfunctions $\phi_N = \exp(-W_N)$.

As an example, let us take $Q_1(z) = 1$ and $P_1(z) = bz$. A particular solution of z is $z(x) = x$. From Eq. (10) one gets

$$W_0(x) = \int^x \frac{bx}{1} dx = \frac{bx^2}{2} + \text{const}. \quad (19)$$

The integration constant is to be determined by normalization of the wave function. For $\phi_0 = \exp(-W_0)$ to be square-integrable, one must assume $b > 0$. The BAE (15) are:

$$bx_k - \sum_{l \neq k} \frac{1}{x_k - x_l} = 0, \quad k = 1, \dots, N, \quad (20)$$

and the potential is

$$V_0 = b^2 x^2 - b, \quad \Delta V_N = -2Nb. \quad (21)$$

This leads to the Schrödinger equation:

$$\left(-\frac{d^2}{dx^2} + b^2 x^2 \right) e^{-W_N} = b(2N+1)e^{-W_N}. \quad (22)$$

This system is just the well-known simple harmonic oscillator.

We note here that by rescaling $\sqrt{b}x_k \rightarrow x_k$, Eq. (20) will have $b = 1$. The resulted equations are the equations that determine the zeros of the Hermite polynomials $H_N(x)$ as found by Stieltjes [31, 41, 42]. Hence we have reproduced the well known wave functions for the harmonic oscillator, namely, $\phi_N = \exp(-W_N) \sim \exp(-bx^2/2)H_N(\sqrt{b}x)$.

B. Type 1 quasi-exactly solvable cases: $m = 2$

Next we consider $P_2(z) = A_2 z^2 + A_1 z + A_0$. By a similar argument we obtain

$$\Delta V_N = -2A_2 N z - 2A_2 \sum_{k=1}^N z_k - 2A_1 N. \quad (23)$$

The Schrödinger can be written as

$$\left(-\frac{d^2}{dx^2} + W_0'^2 - W_0'' - 2A_2 N z \right) e^{-W_N} = 2 \left(A_2 \sum_{k=1}^N z_k + A_1 N \right) e^{-W_N}. \quad (24)$$

Unlike the previous case, now N not only appears in an additive constant term but also in the term with z , and the set of roots z_k 's appear in the additive term. This system is the so-called type 1 QES models. Type 1 QES models classified as class VI in [3] belong to this category.

A well-known example is the sextic oscillator, the simplest QES model of this type [1]. In our prepotential approach, this system is defined by $z(x) = x^2$ and $P_2(z) = 2(az^2 + bz)$. Then $Q_1(z) = 4z$, and

$$W(x) = \int^x \frac{ax^4 + bx^2}{\sqrt{x^2}} dx = \frac{1}{4}ax^4 + \frac{1}{2}bx^2 + \text{const.} \quad (25)$$

Here $a > 0$ to ensure square-integrability of the wave function. The BAE are:

$$2az_k^2 + 2bz_k - 1 - 4 \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0, \quad k = 1, \dots, N, \quad (26)$$

and the potential is

$$V_N = a^2 x^6 + 2abx^4 + [b^2 - (4N+3)a]x^2 - 4a \sum_k z_k - (4N+1)b. \quad (27)$$

It is seen that the QES sextic oscillator can be so easily constructed in the prepotential approach.

C. Type 2 quasi-exactly solvable cases: $m = 3$

We now consider cases with $m = 3$ with $P_3(z) = A_3 z^3 + A_2 z^2 + A_1 z + A_0$. Eq. (11) leads to

$$\Delta V_N = -2A_3 N z^2 - 2 \left(A_3 \sum_{k=1}^N z_k + A_2 N \right) z - 2A_3 \sum_{k=1}^N z_k^2 - 4A_2 \sum_{k=1}^N z_k - 2A_1 N. \quad (28)$$

Now N appears in z and z^2 terms, and also in an additive constant term. The roots z_k 's now not only appear in the additive term but also in the term with z . This is a type 2 QES model.

A simple example of this type is given by the defining relations $z(x) = x$ and $P_3(z) = az^3 + bz$. The prepotential is $W_0 = ax^4/4 + bx^2/2$, which is exactly the same as that for the sextic oscillator discussed in the last section. The two models differ only in the choice of $z(x)$. The corresponding V_N is

$$V_N = a^2x^6 + 2abx^4 + [b^2 - (2N+3)a]x^2 - 2a\left(\sum_{k=1}^N x_k\right)x - 2a\sum_{k=1}^N x_k^2 - (2N+1)b. \quad (29)$$

This is a new QES model.

It is now easy to see that, for $m \geq 4$, not only will N appear in more terms involving powers of z , but also the set of roots z_k 's. This will give rise to new general types of QES systems as mentioned in Sect. II.

IV. EXAMPLES WITH $z'^2(x) = Q_2(z)$

We now come to cases where $z'^2(x) = Q_2(z)$ with $q_2 \neq 0$. Here $z(x)$ is again some sinusoidal coordinates, which include the exponential, hyperbolic and trigonometric functions. Construction of models proceeds as before. By taking appropriate $P_m(z)$, one can reconstruct class I, II and X QES models in [3], and some of the ES models listed in [29], namely, the Morse potential, the Scarf I and II potentials, and the Pöschl-Teller potential. We will not bore the reader by going through all the cases here. Instead we shall briefly discuss the Morse potential, as we would like to show how easy its potential, eigenvalues and eigenfunctions are constructed, and to compare this construction with another construction based on a different choice of the prepotential to be discussed in the next section.

Suppose we take $Q_2(z) = \alpha^2 z^2$, and choose a solution $z(x) = \exp(\alpha x)$ (henceforth α is taken as a positive real constant). Let $P_1(z) = \alpha(Az - B)$ ($A, B > 0$ real constants). The parametrization is chosen such that the form of the Morse potential given in [29] is recovered. From Eq. (10) we have $W'_0 = A - B \exp(-\alpha x)$. Hence

$$V_N = A^2 - B(2A + \alpha)e^{-\alpha x} + B^2e^{-2\alpha x} - [A^2 - (A - N\alpha)^2], \quad (30)$$

with z_k 's satisfying the BAE

$$Az_k - B - \frac{\alpha}{2}z_k - \alpha \sum_{l \neq k} \frac{z_k^2}{z_k - z_l} = 0. \quad (31)$$

This is the ES shape-invariant Morse potential listed in [29]. Taking the first three terms in Eq. (30) as the traditional Morse potential, the eigenvalues are given by $A^2 - (A - N\alpha)^2$, in agreement with the result given in [29]. The wave functions are

$$\phi_N(x) \sim \exp\left(-Ax - \frac{B}{\alpha}e^{-\alpha x}\right) \prod_{k=1}^N (z - z_k), \quad z = e^{\alpha x}. \quad (32)$$

Let us make an interesting observation here. From Eq. (11) it is obvious that only the coefficients q_2 in $Q_2(z)$ and p_1 in $P_1(z)$ will enter the expression of the eigenvalues, namely, $N(2p_1 - q_2N)$. Now for the Scarf II and the Pöschl-Teller potential (both with $Q_2(z) = \alpha^2(1 + z^2)$), $q_2 = \alpha^2$ is the same as that in the Morse potential, whereas for the Scarf I potential (with $Q_2(z) = \alpha^2(1 - z^2)$) there is a sign difference. So if we choose $P_1(z)$ with the same p_1 for all these cases, e.g. $P_1(z) = \alpha Az + p_0$, then we would expect that the Morse, Scarf II and Pöschl-Teller potentials would have the same set of eigenvalues $(A^2 - (A - N\alpha)^2)$, while the Scarf I potential has a different set of eigenvalues differing by a sign in some parameter $((A + N\alpha)^2 - A^2)$. This is in fact the case [29]. The prepotential approach presented here gives a very simple and direct explanation of why this is so.

V. PREPOTENTIALS FOR SYSTEMS WITH SINGULARITY $z = 0$

Now we would like to discuss a possible generalization of the N -th order prepotential in Eq. (4) for quantum systems defined on a half-line (e.g. $x \in (0, \infty)$) with singularity at the origin. For such systems, the wave functions may acquire a prefactor x^p , where p is usually some non-negative positive number, in order to account for the asymptotic behavior at the origin. This observation motivates a possible generalization of the zero-th order prepotential $W_0(x)$

to $\widetilde{W}_0(x) \equiv W_0(x) - p \ln |z|$ for the ground state $\phi_0(x) = \exp(-\widetilde{W}_0(x))$. Here W_0 is a regular function of x as before. Eq. (4) becomes

$$W_N(x, \{z_k\}) = W_0(x) - p \ln |z| - \sum_{k=1}^N \ln |z(x) - z_k|, \quad z_k \neq 0. \quad (33)$$

For the moment p is a free parameter.

With the prepotential (33), the potential $V_N = W_N'^2 - W_N''$ has the form $V_0 + \Delta V_N$ where V_0 and ΔV_N are given by

$$\begin{aligned} V_0 &= \widetilde{W}_0'^2 - \widetilde{W}_0'' \\ &= W_0'^2 - W_0'' - 2 \left(W_0' z' - \frac{z''}{2} \right) \frac{p}{z} + p(p-1) \left(\frac{z'}{z} \right)^2 \end{aligned} \quad (34)$$

$$\Delta V_N = -2 \left(W_0' z' - \frac{z''}{2} \right) \sum_{k=1}^N \frac{1}{z - z_k} + z'^2 \left(\sum_{\substack{k,l \\ k \neq l}} \frac{1}{(z - z_k)(z - z_l)} + \sum_{k=1}^N \frac{2p}{z(z - z_k)} \right). \quad (35)$$

Again, the system is completely defined by the choice of z'^2 and W_0 . However, the presence of terms with the parameter p changes qualitatively the sufficient conditions discussed in Sect. II and opens up new possibilities.

As in previous sections, for definiteness, we shall confine our discussions here to $z'^2 = Q_2(z) \equiv q_2 z^2 + q_1 z + q_0$ and $W_0' z' = P_m(z)$. With these choices, the potential V_N is given by

$$\begin{aligned} V_N &= W_0'^2 - W_0'' - p \left(2P_m(z) - q_1 \left(p - \frac{1}{2} \right) \right) \frac{1}{z} + p(p-1) q_0 \frac{1}{z^2} + q_2 (N+p)^2 \\ &\quad - 2 \sum_{k=1}^N \frac{1}{z - z_k} \left\{ P_m(z) + \frac{pq_0}{z} - \left(p + \frac{1}{2} \right) q_2 z_k - \left(p + \frac{1}{4} \right) q_1 - \sum_{l \neq k} \frac{Q_2(z_k)}{z_k - z_l} \right\}. \end{aligned} \quad (36)$$

The corresponding Bethe ansatz equations are

$$P_m(z_k) + \frac{pq_0}{z_k} - \left(p + \frac{1}{2} \right) q_2 z_k - \left(p + \frac{1}{4} \right) q_1 - \sum_{l \neq k} \frac{Q_2(z_k)}{z_k - z_l} = 0. \quad (37)$$

It is clear that Eqs. (36) and (37) reduce to (11) and (15) when $p = 0$.

If $q_0 = 0$ (i.e., $z = 0$ is a zero of z'^2) and $m \leq 1$ one may obtain an ES model. For example, if we take $Q_1(z) = \alpha z$ and $P_1(z) = Az$, then the three-dimensional oscillator listed in [29] is recovered. But when $q_0 \neq 0$, the presence of the $pq_0/z(z - z_k)$ term in Eq. (36) will give rise to a term $(\sum_k z_k)/z$ in V_N , and hence the potential (36) represents a QES system even if $m \leq 1$, in contrast to the cases discussed in previous sections. For instance, if we take $z'^2 = 1$ and $P_m(z) = P_1(z) = a + bz$, this will produce an ES model if $p = 0$, which is the shifted oscillator $V_N = (bx+a)^2 - (2N+1)b$ (note that the domain of x changes from the half-line to the full line). But for general p , the system is a type 2 QES system classified as the class VIII system in [3].

However, it is possible that if p assumes certain value, the nature of the system could be qualitatively changed, such as the domain of the variable may change from the half-line to the full line (as in the case of the shifted oscillator mentioned in the previous paragraph), or a QES system becomes an ES one. We shall illustrate these situations with two examples below.

A. Sextic oscillator again

Following Sect. III(B), we take $Q_2(z) = 4z$ and $P_m(z) = 2(az^2 + bz)$. These lead to $z = x^2$ and $W_0(x) = ax^4/4 + bx^2/2 + c$, with real constants $a > 0$, b and c . The BAE and V_N are

$$2az_k^2 + 2bz_k - (4p+1) - 4 \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0, \quad k = 1, \dots, N, \quad (38)$$

and

$$\begin{aligned} V_N &= a^2 x^6 + 2abx^4 + [b^2 - (4N+4p+3)a] x^2 \\ &\quad + 4p \left(p - \frac{1}{2} \right) \frac{1}{x^2} - 4a \sum_k z_k - (4N+4p+1)b. \end{aligned} \quad (39)$$

In general, this model is a QES system on the half-line. But if $p = 0$ or $p = 1/2$ the $1/x^2$ term will be absent, and the system is just the sextic oscillator on the full line. The domain is extended. The case $p = 0$ with symmetric wave functions was discussed before, which is the class VII QES model in [3], and the case $p = 1/2$ with anti-symmetric wave functions was discussed in [1, 4, 5].

B. Morse potential again

Now we consider the situation which will include the ES Morse potential as a special case.

We take $Q_2(z) = \alpha^2 z^2$ and $P_m(z) = P_2(z) = \alpha^2 z^2/2 - \alpha A z$. As a special case, we choose the solution $z = \exp(-\alpha x)$ and $W'_0(x) = A - Bz$ with $B \equiv \alpha/2$. As in Sect. IV, the parametrization is chosen such that when p assumes special value, the system becomes the Morse potential given in [29]. The potential is

$$V_N = A^2 - B(2A + \alpha)e^{-\alpha x} + B^2 e^{-2\alpha x} - 2p\alpha \left(\frac{\alpha}{2}z - A\right) + \alpha^2(N + p)^2 - 2 \sum_{k=1}^N \frac{1}{z - z_k} \left\{ \frac{\alpha^2}{2}z^2 - \alpha A z - \alpha^2 \left(p + \frac{1}{2}\right) z_k - \sum_{l \neq k} \frac{\alpha^2 z_k^2}{z_k - z_l} \right\}. \quad (40)$$

If we choose z_k 's to satisfy the BAE (37)

$$\frac{\alpha}{2}z_k - A - \alpha \left(p + \frac{1}{2}\right) - \alpha \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0, \quad k = 1, 2, \dots, N, \quad (41)$$

then we arrive at the potential

$$V_N = A^2 - B(2A + \alpha)e^{-\alpha x} + B^2 e^{-2\alpha x} - \alpha^2(N + p)z - \alpha^2 \sum_k z_k + 2\alpha A(N + p) + \alpha^2(N + p)^2. \quad (42)$$

The term $-\alpha^2 \sum_k z_k$ can be simplified using Eq. (41) as

$$-\alpha^2 \sum_k z_k = -\alpha[2A + (2p + 1)\alpha]N - \alpha^2 N(N - 1). \quad (43)$$

In general this potential defines a type 1 QES system, as N appears in the first power term of z . This system is not listed in [3]. But if $p = 0$, then the domain of x changes from the half-line to the full line, and the system becomes that classified as class I in [3].

It is also obvious that if $p = -N$, V_N becomes Eq. (30), and the system is the ES Morse potential. Let us recast Eq. (41) into the following form (taking $p = -N$):

$$\sum_{l \neq k} \frac{1}{z_k - z_l} + \frac{\gamma/2}{z_k} = \frac{1}{2}, \quad k = 1, 2, \dots, N, \quad (44)$$

where

$$\gamma \equiv 2\frac{A}{\alpha} - 2N + 1. \quad (45)$$

Eq. (44) is just the set of equations that determines the zeros of the associated Laguerre polynomials $L_N^{\gamma-1}(z)$, i.e., $L_N^{\gamma-1}(z_k) = 0$ [31, 42]. Hence the eigenfunctions $\phi_N = \exp(-W_N)$ are

$$\begin{aligned} \phi_N(x) &= \exp\left(-Ax - \frac{B}{\alpha}e^{-\alpha x}\right) z^{-N} \prod_{k=1}^N (z - z_k) \\ &= z^{s-N} e^{-\frac{\gamma}{2}z} L_N^{2(s-N)}(z), \quad s \equiv \frac{A}{\alpha}, \quad z = e^{-\alpha x}, \end{aligned} \quad (46)$$

as given in [29].

Compared with the discussion in Sect. IV, it is interesting to see that the Morse potential can be constructed with two different prepotentials. Now one may wonder if the results are consistent, as the wave functions and the BAE look rather different in these two constructions. Below we would like to show that they are indeed the same.

Let us rewrite Eq. (32) as

$$\phi_N(x) \sim \exp\left(-Ax - \frac{B}{\alpha}e^{-\alpha x}\right) z^N \prod_{k=1}^N (z^{-1} - z_k^{-1}), \quad z = e^{\alpha x}. \quad (47)$$

We note that in Sect. IV the variable $z = \exp(\alpha x)$ is reciprocal to the variable $z = \exp(-\alpha x)$ in this subsection. Hence when one makes the change $z \rightarrow 1/z$ and $z_k \rightarrow 1/z_k$ in Eq. (47), one arrives at Eq. (46). Now one needs only to show that the same transformation in z and z_k maps the BAE (31) to Eq. (41).

Making the change $z_k \rightarrow 1/z_k$ in Eq. (31) leads to

$$A - \frac{\alpha}{2} - Bz_k - \alpha \sum_{l \neq k} \frac{z_l}{z_l - z_k} = 0. \quad (48)$$

Writing $z_l = (z_l - z_k) + z_k$ in the numerator of the last term in Eq. (48) and recalling that $B = \alpha/2$, we arrive at

$$\frac{\alpha}{2} z_k - A - \alpha \left(-N + \frac{1}{2}\right) - \alpha \sum_{l \neq k} \frac{z_k}{z_k - z_l} = 0, \quad k = 1, 2, \dots, N. \quad (49)$$

This is simply Eq. (41) with $p = -N$. Thus we have shown that the wave functions and the BAE obtained in the two constructions are the same.

VI. SYSTEMS DEFINED ON FINITE INTERVALS

Finally, we consider systems defined on a finite interval. Suppose the potential of a system is singular at $z = a_1$ and a_2 , where a_1 and a_2 are two real parameters (assuming $a_2 > a_1$). Generalizing the discussions in the last section, it is plausible to assume for such system a prepotential of the form

$$W_N(x, \{z_k\}) = W_0(x) - p_1 \ln |z - a_1| - p_2 \ln |z - a_2| - \sum_{k=1}^N \ln |z(x) - z_k|, \quad z_k \neq a_1, a_2, \quad (50)$$

where p_1 and p_2 are two real positive parameters. The wave function has the form

$$\phi_N \sim \exp(-W_0(x))(z - a_1)^{p_1}(z - a_2)^{p_2} \prod_{k=1}^N (z - z_k) \quad (51)$$

Now Eqs. (34) and (35) are generalized to

$$V_0 = W_0'^2 - W_0'' - 2 \left(W_0' z' - \frac{z''}{2} \right) \left(\frac{p_1}{z - a_1} + \frac{p_2}{z - a_2} \right) + z'^2 \left[\frac{p_1(p_1 - 1)}{(z - a_1)^2} + \frac{p_2(p_2 - 1)}{(z - a_2)^2} + \frac{2p_1 p_2}{(z - a_1)(z - a_2)} \right], \quad (52)$$

$$\Delta V_N = -2 \left(W_0' z' - \frac{z''}{2} \right) \sum_{k=1}^N \frac{1}{z - z_k} + z'^2 \left[\sum_{\substack{k,l \\ k \neq l}} \frac{1}{(z - z_k)(z - z_l)} + 2 \left(\frac{p_1}{z - a_1} + \frac{p_2}{z - a_2} \right) \sum_{k=1}^N \frac{1}{z - z_k} \right]. \quad (53)$$

Once again, the system is completely defined by the choice of z'^2 and W_0 . The analysis of the solvability of the system is more complicated than before, but the principle is the same. For polynomial z'^2 and $W_0' z'$ the systems are new QES models in general.

As before let us take z'^2 to be at most quadratic in z . A situation of interest is that in which a_1 and a_2 are the two real zeros of $Q_2(z)$, i.e., if $z'^2 = A(z - a_1)(z - a_2)$ where A is real. In this case the sufficient conditions for ES and QES models are the same as before, namely, one could get ES system if $m \leq 1$, and QES otherwise. But just as the Morse potential discussed in the last section, the QES system could become an ES one if p_1 and p_2 take on special values. The ES Rosen-Morse I and II potentials and the Eckart potential in [29] are such cases.

We shall illustrate the construction of a QES model first discussed in [28] (see also [37]). This model was not listed in [3] but is found to be also related to $sl(2)$ algebra [37]. The function $z(x)$ is taken to be $z(x) = \sin^2 x$. This is a solution of $z'^2 = 4z(1 - z)$, and hence $a_1 = 0$ and $a_2 = 1$, and $0 < x < \pi/2$. In order to obtain a type 1 QES model we should choose W_0 such that $W_0'z'$ is of the second degree in z . Let us take $P_2(z) = 4az(z - 1)$, where a is real. This gives a solution $W_0(x) = a \cos 2x/2$. With the chosen $Q_2(z)$ and $P_2(z)$, we obtain from Eq. (52) and (53) the following QES potential

$$\begin{aligned} V_N(x) = & a^2 \sin^2 2x + 2a \cos 2x + 2(a \sin^2 2x + \cos 2x) \left(\frac{p_1}{\sin^2 x} - \frac{p_2}{\cos^2 x} \right) \\ & + 4p_1(p_1 - 1) \cot^2 x + 4p_2(p_2 - 1) \tan^2 x \\ & - 8aN \sin^2 x - 8a \sum_{k=1}^N z_k - 4N[N - 2(a - p_1 - p_2)] - 8p_1p_2, \end{aligned} \quad (54)$$

where the z_k 's satisfy the BAE ($k = 1, 2, \dots, N$):

$$4az_k^2 - 2(2(a - p_1 - p_2) - 1)z_k - 1 - 4z_k(1 - z_k) \sum_{l \neq k} \frac{1}{z_k - z_l} = 0. \quad (55)$$

The forms of V_N and BAE in [37] for this model is regained by identifying $p_1 = c/2 - b/4$ and $p_2 = b/4$, where b and c are the parameters used in that paper.

VII. SUMMARY

We have discussed exact and quasi-exact solvability of the schrödinger equation based on the approach of prepotential. Three types of N -th order prepotentials are described which cover most of the known ES and QES models, and which are capable of generating new ones. It is shown that the choice of coordinate transformation $z'^2(x)$ and the zero-th order prepotential $W_0(x)$ completely determine the form of V_N . General conditions on the choice of z'^2 and $W_0(x)$ for exact and quasi-exact solvabilities were described. These conditions were illustrated by various examples. The prepotential approach is quite elegant in itself. Moreover, it allows easy extensions of ES and QES theory to systems with multi-component wave functions, such as the Pauli and Dirac equations, and to the Fokker-Planck equations, as prescribed in [14, 21, 22, 37].

In this work we have confined our discussions to ES and QES models involving a change of coordinates to the so-called sinusoidal coordinates. These are coordinates $z(x)$ which satisfy $z'^2 = Q_2(z)$, or $z'' = R_0(z)$ or $R_1(z)$. These coordinates cover most of the known ES and QES models. The examples we presented here by no means exhaust all possible sinusoidal coordinates. Other QES cases admitted by certain sinusoidal transformations need be studied.

But as discussed in Sect. II, to construct a type 1 QES system it is also possible to choose z'^2 to be a cubic polynomial, i.e. $z'^2 = Q_3(z)$. This lies beyond the transformations by sinusoidal coordinates. Such transformations are less studied in the literature. However, they could give rise to new QES systems, and deserve further study. In fact, two $sl(2)$ -based QES cases in [3], namely, class IV and V, require $n = 3$ and $m = 1$ and 3, respectively.

Only three types of N -th order prepotentials were discussed here, namely, prepotentials for systems defined on the whole line, on the half-line, and on a finite interval. Generalizing to several singularities is straightforward.

We have only considered cases for which $W_0'z'$ and z'^2 are polynomials in z . It is interesting to consider other possibilities for these two defining functions.

Finally, it is also interesting to extend the present approach to systems with non-Hermitian Hamiltonians admitting real spectra. A preliminary attempt was presented in [43], where some non-Hermitian QES Hamiltonians, including that given in [44], were generated by making some coefficients in $P_m(z)$ complex. However, a full treatment of this kind of systems usually requires one to extend the basic variable x to the complex plane [45, 46]. Hence a better understanding of the prepotential approach to non-Hermitian Hamiltonians is needed.

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- [1] A.V. Turbiner and A.G. Ushveridze, Phys. Lett. **A126** (1987) 181.
 - [2] A.G. Turbiner, Sov. Phys. JETP **67** (1988) 230.
 - [3] A.V. Turbiner, Comm. Math. Phys. **118** (1988) 467.
 - [4] A.G. Ushveridze, Sov. Phys.-Lebedev Inst. Rep. **2** (1988) 50; 54.
 - [5] A.G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics* (IOP, Bristol, 1994).
 - [6] N. Kamran and P.J. Olver, J. Math. Anal. Appl. **145** (1990) 342.
 - [7] A. González-López, N. Kamran and P.J. Olver, Comm. Math. Phys. **153** (1993) 117.
 - [8] A. González-López, N. Kamran and P.J. Olver, Contemp. Math. **160** (1994) 113.
 - [9] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. **126** (1989) 347.
 - [10] A.V. Turbiner, Contemp. Math. **160** (1994) 263.
 - [11] G. Post and A.V. Turbiner, Russ. J. Math. Phys. **3** (1995) 113.
 - [12] M.A. Shifman, Int. J. Mod. Phys. **A4** (1989) 2897; 3305.
 - [13] M.A. Shifman, Contemp. Math. **160** (1994) 237.
 - [14] C.-L. Ho and P. Roy, J. Phys. **A36** (2003) 4617.
 - [15] Y. Brihaye and P. Kosinski, Mod. Phys. Lett. **A13** (1998) 1445.
 - [16] Y. Brihaye and A. Nininahazwe, Mod. Phys. Lett. **A20** (2005) 1875.
 - [17] M. Znojil, Mod. Phys. Lett. **A14** (1999) 863.
 - [18] C.-L. Ho and V.R. Khalilov, Phys. Rev. **A61** (2000) 032104.
 - [19] C.-M. Chiang and C.-L. Ho, J. Math. Phys. **43** (2002) 43.
 - [20] C.-M. Chiang and C.-L. Ho, Mod. Phys. Lett. **A20** (2005) 673.
 - [21] C.-L. Ho and P. Roy, Ann. Phys. **312** (2004) 161.
 - [22] C.-L. Ho, Ann. Phys. **321** (2006) 2170.
 - [23] A. Schulze-Halberg, Chinese J. Phys. **42** (2004) 117; 234.
 - [24] H.-T. Cho and C.-L. Ho, J. Phys **A40** (2007) 1325.
 - [25] For a first systematic discussion of the use of Bethe ansatz equations, see e.g., M. Gaudin, *La Fonction d'Onde de Bethe* (Masson, Paris, 1983).
 - [26] G. Post and A. Turbiner, Russian J. Math. Phys. **3** (1995) 113.
 - [27] N. Kamran, R. Milson and P.J. Olver, Invariant modules and the reduction of nonlinear partial differential equations to dynamical systems (1999). solv-int/9904014.
 - [28] R. Sasaki and K. Takasaki, J. Phys. **A34** (2001) 9533.
 - [29] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267.
 - [30] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer-Verlag, Berlin, 1996).
 - [31] E. Corrigan and R. Sasaki, J.Phys. **A35** (2002) 7017.
 - [32] A.J. Bordner, N.S. Manton and R. Sasaki, Prog. Theor. Phys. **103** (2000) 463.
 - [33] S.P. Khastgir, A.J. Pocklington and R. Sasaki, J. Phys. **A33** (2000) 9033.
 - [34] S. Odake and R. Sasaki, J. Phys. **A35** (2002) 8283.
 - [35] I. Loris and R. Sasaki, J. Phys. **A37** (2004) 211.
 - [36] R. Sasaki and K. Takasaki, J. Math. Phys. **47** (2006) 012701.
 - [37] C.-L. Ho and R. Sasaki, Quasi-exactly solvable Fokker-Planck equations, Tamkang preprint and Yukawa Institute (Kyoto Univ.) Report No.Y ITP-07-21 (to appear in Ann. Phys.) arXiv:0705.0863.v2.
 - [38] E. Schrödinger, Proc. Roy. Irish Acad. **A46** (1940) 9.
 - [39] L. Infeld and T.E. Hull, Rev. Mod. Phys. **23** (1951) 21.
 - [40] S. Odake and R. Sasaki, J. Math. Phys. **47** (2006) 102102.
 - [41] T.J. Stieltjes, "Sur quelques théorèmes d'algèbre", Compte Rendus **100** (1885) 439-440. In Stieltjes' approach, the zeros of classical polynomials can be interpreted as the equilibrium positions of a related electrostatic system. Such analogy was rediscovered in [4].
 - [42] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publications Vol. 23 (Amer. Math. Soc., New York, 1939).
 - [43] C.-L. Ho, Prepotential approach to exact and quasi-exact solvabilities of Hermitian and non-Hermitian Hamiltonians. Talk presented at "Conference in Honor of CN Yang's 85th Birthday", 31 Oct - 3 Nov, 2007, Singapore. arXiv:0801.0944 [hep-th].
 - [44] C.M. Bender and S. Boettcher, J. Phys. **A31** (1998) L273.
 - [45] C.M. Bender and S. Boettcher, Phys. Rev. Lett. **80** (1998) 5234.
 - [46] C.M. Bender, Contemp.Phys. **46** (2005) 277.